ON THE SOLUTION OF NONLINEAR BOUNDARY VALUE PROBLEMS OF THE THEORY OF ELASTICITY BY A METHOD OF TRANSFORMATION TO AN INITIAL VALUE CAUCHY PROBLEM

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At present the solution of the nonlinear problems of shell stability is carried out, in the main, with the aid of direct variational methods. However, the application of these methods to the higher approximations is greatly impeded because of the awkwardness of the finite-dimensional equations which are obtained. In this paper a method of transformation to an initial value problem is used which permits, with the aid of an electronic digital computer, the determination of any functional of the solution when the problem is solved by a direct variational method in the higher approximations.

The application of the method is illustrated by the problems of uniform pressure or a concentrated load applied to a sherical cap.

The present method may be treated as a variation of the step-by-step method used previously in [1 and 2]. Some recommendations are given for the elaboration of the method.

1. We write the equations of the deformed state of the shell symbolically in the form

$$A_i(u, v, w, p) = 0 \qquad (i = 1, 2, 3) \tag{1.1}$$

where u, v and w are the displacements and p is the loading parameter. Let us assume that we must determine some functional $\Phi(u, v, w)$, which is finally a function of the loading parameter p. In an approximate solution of the system (1.1) by any direct variational method, we arrive at some system of finite-dimensional equations which are, in general, transcendental. Thus, for example, in using the Bubnov-Galerkin method, the displacement vector $\mathbf{a}$ is approximated by aggregates of the form

$$\mathbf{a}_n = \sum_{k=1}^{n} C_{nk}\mathbf{b}_k \tag{1.2}$$

where the C_{nk} are constant and $\mathbf{b}_k$ is a system of vectors which is complete for the given problem. The Bubnov-Galerkin procedure yields a system of equations of the form

$$B_r(C_{nk}, p) = 0 \qquad (r = 1, \ldots, n) \tag{1.3}$$

to determine the C_{nk}.

In this same approximation we may consider the desired functional Φ as a function of the parameters C_{nk}. To determine the C_{nk} and the functional Φ we make use of the idea set forth in [3 to 6] of transformation to an initial value problem for ordinary differential equations, altering it somewhat in view of the fact that the chief aim is the determination of the value of the functional Φ. Below, some practical recommendations will also be given for the use of this device.

Let us assume that the solution of the system (1.3) is known for some value p_0 and, therefore, that the value Φ_0 of the functional is known. We differentiate the system (1.3) through with respect to p and obtain

$$\frac{\partial B_r}{\partial p} + \sum_{k=1}^{n} \frac{\partial B_r}{\partial C_{nk}} \frac{dC_{nk}}{dp} = 0 \qquad (r = 1, \ldots, n) \tag{1.4}$$

We now adjoin Equation

$$\frac{d\Phi}{dp} - \sum_{k=1}^{n} \frac{\partial \Phi}{\partial C_{nk}} \frac{dC_{nk}}{dp} = 0 \tag{1.5}$$

to the system (1.4).

The system (1.4), (1.5) can be considered as a system of linear, ordinary equations for Φ and C_{nk}. Using the initial data, we can solve the initial value problem for this system and find the solution for a sufficiently wide range of variation of the parameter p. The integration of the system (1.4), (1.5) may be carried out by any method, e.g. by the Runge-Kutta method. Some inconvenience arises in this connection because the system (1.4),(1.5) is not solved for the derivatives. However, this difficulty is easily circumvented. The following circumstances may be obstacles to the use of the present method.

1. Any of the derivatives dC_{nk} / dp, $d\Phi / dp$ go to infinity.
2. Singular points are presented on the curves $C_{nk}(p)$, $\Phi(p)$.

The first obstacle may be avoided if a new independent variable, say Φ, is used instead of p. Then the system (1.4),(1.5) assumes the form

$$\frac{\partial B_r}{\partial p} \frac{dp}{d\Phi} + \sum_{k=1}^{n} \frac{\partial B_r}{\partial C_{nk}} \frac{dC_{nk}}{d\Phi} = 0 \tag{1.6}$$

$$\sum_{k=1}^{n} \frac{\partial \Phi}{\partial C_{nk}} \frac{dC_{nk}}{d\Phi} - 1 = 0 \tag{1.7}$$

Use of the system (1.4),(1.5) or (1.6),(1.7) in general makes it possible to traverse the entire $\Phi - p$ curve if there are no singular points on it.

In this connection, it is possible in the computer program to provide for automatic transfer from one system to the other in those regions where the use of one of them turns out to be impractical. Such transfer can be accomplished, for example, by using $d\Phi/dp$ as a criterion, applying the system (1.4),(1.5) when $|d\Phi/dp| < 1$ and (1.6),(1.7) when $|d\Phi/dp| > 1$. It is possible, however, to avoid any transitions from one system to another is a more complicated parameter σ is introduced, σ being defined by the relation

$$\sigma = \int_0^p \left[1 + \left(\frac{d\Phi}{dp}\right)^2\right]^{1/2} dp \tag{1.8}$$

The parameter σ is the arc length along the curve $\Phi = \Phi(p)$; the system which corresponds has the form

$$\frac{\partial B_r}{\partial p}\frac{dp}{d\sigma} + \sum_{k=1}^{n}\frac{\partial B_r}{\partial C_{nk}}\frac{dC_{nk}}{d\sigma} = 0, \quad \sum_{k=1}^{n}\frac{\partial \Phi}{\partial C_{nk}}\frac{dC_{nk}}{d\sigma} = \frac{d\Phi}{d\sigma}, \quad \frac{dp}{d\sigma} = \left[1 - \left(\frac{d\Phi}{d\sigma}\right)^2\right]^{1/2} \tag{1.9}$$

The system (1.9) now contains no derivatives which can go to infinity and may, therefore, be integrated up to arbitrary large values of the parameter σ, provided that no singular points are encountered along the way. In the case of a singular point, a preliminary investigation of the character of this point is required, and after this it is necessary to devise methods of avoiding it. The same method can also be used when the system (1.3) depends not on a single parameter p, but on several, for example, on two

$$B_r(C_{nk}, p_1, p_2) = 0 \qquad (r = 1, \ldots, n) \tag{1.10}$$

We differentiate the system (1.10) with respect to p_1 and p_2 successively. As a result we obtain

$$\sum_{k=1}^{n}\frac{\partial B_r}{\partial C_{nk}}\frac{\partial^2 C_{nk}}{\partial p_1 \partial p_2} + \sum_{k=1}^{n}\sum_{l=1}^{n}\frac{\partial^2 B_r}{\partial C_{nk}\partial C_{nl}}\frac{\partial C_{nk}}{\partial p_1}\frac{\partial C_{nl}}{\partial p_2} + \sum_{k=1}^{n}\frac{\partial^2 B_r}{\partial p_1 \partial C_{nk}}\frac{\partial C_{nk}}{\partial p_2} + \frac{\partial^2 B_r}{\partial p_1 \partial p_2} = 0 \tag{1.11}$$

$$\frac{\partial^2 \Phi}{\partial p_1 \partial p_2} - \sum_{k=1}^{n}\frac{\partial \Phi}{\partial C_{nk}}\frac{\partial^2 C_{nk}}{\partial p_1 \partial p_2} - \sum_{k=1}^{n}\frac{\partial^2 \Phi}{\partial C_{nk}\partial C_{nl}}\frac{\partial C_{nk}}{\partial p_1}\frac{\partial C_{nl}}{\partial p_2} = 0 \tag{1.12}$$

The system (1.11),(1.12) can be regarded as a system of partial differential equations in C_{nk} and Φ. If some boundary conditions are adjoined to it, then a solution for Φ and C_{nk} as functions of p_1 and p_2 can be found. If the values of C_{nk} and Φ are known on the axes $p_1 = 0$, $p_2 = 0$, then in this case the system is a Goursat problem (a set of partial differential equations with characteristic data). Solving the problem by any numerical method, e.g. by finite differences, we find the roots of the system (1.10) as functions of p_1 and p_2.

2. Let us apply the considerations given above to the investigation of the nonlinear system of equations which describes the axisymmerric deformation of a spherical cap. The equations of the problem may be taken in the form [1]

$$\rho\psi'' + \psi' - \frac{\psi}{\rho} = \theta\left(\frac{2H}{h}\rho + \frac{1}{2}\theta\right) \qquad \left(\psi = -\frac{T_1 a^2 \rho}{Eh^3}, \ \theta = \frac{a}{h}\theta_1\right) \tag{2.1}$$

$$\rho\theta'' + \theta' - \frac{\theta}{\rho} = -12(1-\mu^2)\psi\left(\frac{2H}{h}\rho + \theta\right) + 6(1-\mu^2)p_0\rho^2\left(p_0 = \frac{pa^4}{Eh^4}\right) \tag{2.2}$$

where ρ is the dimensionless radius, T_1 is the radial stress, h is the shell thickness, and θ_1 is the angle of rotation of a section. The remaining notation is given in Fig.1. The system (2.1),(2.2) will be considered along with boundary conditions which correspond to complete fixity

$$\theta = 0, \qquad \left|\psi' - \mu\frac{\psi}{\rho}\right| = 0 \qquad \text{for } \rho = 1 \qquad (2.3)$$

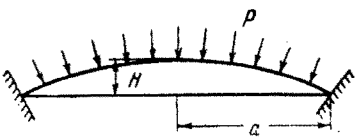

Fig. 1

Let us assume that the problem is to determine the loading curve for the cap, i.e. to find the relation between p_0 and the axial displacement f_1 at the center. It is easy to see that f_1 is determined by the relation

$$f_1 = hf, \qquad f = \int_1^\rho \theta\, d\rho \qquad (2.4)$$

Thus, the integral on the right-hand side of (2.4) plays the role of the functional Φ. In solving the problem by the Bubnov-Galerkin method, we set

$$\theta = \sum_{k=1}^{n} C_{nk}(\rho^{2k+1} - \rho^{2k-1}) \qquad (2.5)$$

From Equation (2.1), taking account of the boundary conditions (2.3), we obtain ψ in the form

$$\psi = \frac{\lambda}{4}\sum_{k=1}^{n} C_{nk}\left[\frac{1}{(k+2)(k+1)}\left(\rho^{2k+3} - \frac{2k+3-\mu}{1-\mu}\rho\right) - \frac{1}{(k+1)k}\times\right.$$
$$\left.\times\left(\rho^{2k+1} - \frac{2k+1-\mu}{1-\mu}\rho\right)\right] + \frac{1}{8}\sum_{k=1}^{n}\sum_{l=1}^{n} C_{nk}C_{nl}\left[\frac{1}{(k+l+2)(k+l+1)}\times\right.$$
$$\times\left(\rho^{2k+2l+3} - \frac{2k+2l+3-\mu}{1-\mu}\rho\right) -$$
$$-\frac{2}{(k+l+1)(k+l)}\left(\rho^{2k+2l+1} - \frac{2k+2l+1-\mu}{1-\mu}\rho\right) +$$
$$\left.+\frac{1}{(k+l)(k+l-1)}\left(\rho^{2k+2l-1} - \frac{2k+2l-1-\mu}{1-\mu}\rho\right)\right] \qquad \left(\lambda = \frac{2H}{h}\right) \qquad (2.6)$$

We substitute (2.5),(2.6) into the left-hand side of (2.2) and require that the expression obtained be ortogonal to $(\rho^{2r+1} - \rho^{2r-1})$ $(r = 1, 2, \ldots, n)$. In this way we obtain the following system of equations for the C_{nk} :

$$\sum_{k=1}^{n}(A_{kr}^{(1)} + \lambda^2 A_{kr}^{(2)})C_{nk} + \lambda\sum_{k=1}^{n}\sum_{l=1}^{n} A_{klr}C_{nk}C_{nl} + \sum_{k=1}^{n}\sum_{l=1}^{n}\sum_{m=1}^{n} A_{klmr}C_{nk}C_{nl}C_{nm} =$$
$$= p_0 A_r \qquad (r = 1, 2, \ldots, n) \qquad (2.7)$$

where the coefficients $A_{kr}^{(1)}, A_{kr}^{(2)}, A_{klr}$ and A_{klmr} are given by Equations

$$A_{kr}^{(1)} = -\frac{4kr}{(1-\mu^2)}\prod_{i=-1}^{1}\frac{1}{k+r+i}, \qquad A_{kr}^{(2)} = -\frac{3}{1-\mu}\prod_{i=1}^{2}\frac{1}{k+i}\prod_{i=1}^{2}\frac{1}{r+i} -$$

$$-3\left[r^2+k^2+3kr+6(r+k)+7\right]\prod_{i=1}^{2}\frac{1}{r+i}\prod_{i=1}^{2}\frac{1}{k+i}\prod_{i=1}^{3}\frac{1}{r+k+i}$$

$$A_{klr} = \frac{3}{1-\mu}\left(\prod_{i=0}^{2}\frac{1}{k+l+i}\prod_{i=1}^{2}\frac{1}{r+i} + 2\prod_{i=0}^{2}\frac{1}{r+l+i}\prod_{i=1}^{2}\frac{1}{k+i}\right) +$$

$$+3\Bigg[\prod_{i=1}^{2}\frac{1}{r+i}\left(\prod_{i=1}^{3}\frac{1}{k+l+r+i} - \prod_{i=0}^{2}\frac{1}{r+l+i}\right) -$$

$$-\prod_{i=0}^{1}\frac{1}{r+i}\left(\prod_{i=0}^{2}\frac{1}{r+l+k+i} - \prod_{i=0}^{2}\frac{1}{k+l+i}\right) +$$

$$+\frac{1}{4}\prod_{i=1}^{2}\frac{1}{k+l+i}\left(\prod_{i=1}^{2}\frac{1}{r+i} - \prod_{i=2}^{3}\frac{1}{k+l+r+i}\right) -$$

$$-\frac{1}{2}\prod_{i=0}^{1}\frac{1}{k+l+i}\left(\prod_{i=1}^{2}\frac{1}{r+i} - \prod_{i=1}^{2}\frac{1}{k+l+r+i}\right) +$$

$$+\frac{1}{4}\prod_{i=-1}^{0}\frac{1}{k+l+i}\left(\prod_{i=1}^{2}\frac{1}{r+i} - \prod_{i=0}^{1}\frac{1}{k+l+r+i}\right)\Bigg]$$

$$A_{krlm} = -\frac{6}{1-\mu}\prod_{i=0}^{2}\frac{1}{k+m+i}\prod_{i=0}^{2}\frac{1}{r+l+i} +$$

$$+\frac{3}{2}\prod_{i=1}^{2}\frac{1}{k+m+i}\left(\prod_{i=1}^{3}\frac{1}{k+m+l+r+i} - \prod_{i=0}^{2}\frac{1}{r+l+i}\right) -$$

$$-3\prod_{i=0}^{1}\frac{1}{k+m+i}\left(\prod_{i=0}^{2}\frac{1}{k+m+l+r+i} - \prod_{i=0}^{2}\frac{1}{r+l+i}\right) + \qquad (2.8)$$

$$+\frac{3}{2}\prod_{i=-1}^{0}\frac{1}{k+m+i}\left(\prod_{i=1}^{1}\frac{1}{k+m+l+r+i} - \prod_{i=0}^{2}\frac{1}{r+l+i}\right)$$

$$A_r = -3\prod_{i=1}^{2}\frac{1}{r+i}$$

From (2.4) it is easy to obtain Expression

$$f = \sum_{k=1}^{n}\frac{1}{2k(k+1)}C_{nk} \qquad (2.9)$$

for the unknown functional Φ .

Choosing f as the independent parameter, we obtain the following system of differential equations from (2.7),(2.9):

$$\sum_{k=1}^{n} \frac{dC_{nk}}{df} \left[(A_{kr}^{(1)} + \lambda^2 A_{kl}^{(2)}) + \lambda \sum_{l=1}^{n} (A_{klr} + A_{lkr}) C_{nl} + \right.$$

$$\left. + \sum_{l=1}^{n} \sum_{m=1}^{n} (A_{klmr} + A_{lkmr} + A_{mklr}) C_{nl} C_{nm} \right] - \frac{dp_0}{df} A_r \Big] = 0 \qquad (2.10)$$

$$\sum_{k=1}^{n} \frac{1}{2k(k-1)} \frac{dC_{nk}}{df} = 1 \qquad (r = 1, \ldots, n) \qquad (2.11)$$

We can take the unstressed state of the shell, in the absence of load, for the initial data, i.e. for $f = 0,\ p_0 = 0,\ C_{nk} = 0$. It is convenient to integrate the system (2.10), (2.11) as long as dp_0/df is not very large. In the contrary case it is expedient to take p_0 as the independent variable and to obtain a system of the form

$$\sum_{k=1}^{n} \frac{dC_{nk}}{dp_0} \left\{ (A_{kr}^{(1)} + \lambda^2 A_{kr}^{(2)}) + \lambda \sum_{l=1}^{n} (A_{klr} + A_{lkr}) C_{nl} + \right.$$

$$\left. + \sum_{l=1}^{n} \sum_{m=1}^{n} (A_{klmr} + A_{lkmr} + A_{mklr}) C_{nl} C_{nm} \right\} - A_r = 0 \qquad (2.12)$$

$$\sum_{k=1}^{n} \frac{1}{2k(k+1)} \frac{dC_{nk}}{dp_0} = \frac{df}{dp_0} \qquad (r = 1, \ldots, n) \qquad (2.13)$$

The systems (2.10),(2.11) and (2.12),(2.13) were integrated by the Runge-Kutta method. A program was written for the "Minsk-12" digital computer. It consisted of a number of subroutines and provided for automatic transfer from one system to the other.

3. Let us examine some results of the computations. Curves of p_0 vs. f were calculated for values of the parameter $\lambda = 0,1,2,3,4,5,6,7,8$. In all cases it was found that the first approximation in the Bubnov-Galerkin method differed greatly from the exact solution. This fact has been observed before in [1, 7 and 8]. It was possible to obtain satisfactory accuracy

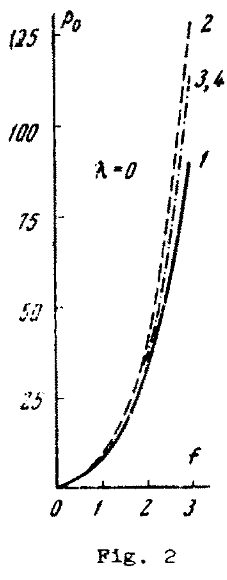

Fig. 2

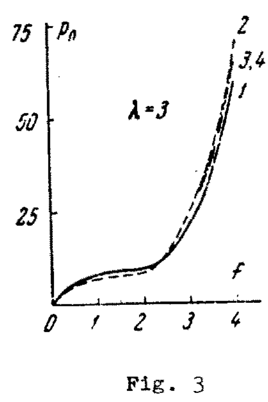

Fig. 3

f	(1)	(2)	(3)	(4)
		$\lambda = 0$		
0.5	3.2756	3.3285	3.3326	3.3330
1.0	8.9227	9.1178	9.1460	9.1467
1.5	18.1126	20.2541	20.1943	20.1830
2.0	33.8168	40.5438	39.4602	39.4351
2.5	57.8067	74.8477	69.7676	69.9035
		$\lambda = 3$		
0.4	5.0582	4.9889	4.9827	4.9818
1.0	8.0227	7.4135	7.3813	7.3798
1.6	8.6214	8.0737	8.0633	8.0635
2.0	9.4740	9.2199	9.2144	9.2226
2.8	17.0395	17.0768	17.4199	17.4828
		$\lambda = 6$		
0.4	16.3725	17.5066	17.6220	17.6249
1.0	29.3655	25.5967	24.9164	24.8500
2.6	21.4384	16.5576	17.4514	17.5085
3.6	8.0154	14.7784	15.3213	15.0961
5.4	28.6860	28.4528	31.1870	32.7468
		$\lambda = 8$		
0.4	28.7915	34.2036	35.0114	35.0591
1.0	55.7846	50.5424	48.6588	48.3647
2.0	66.4263	36.2654	33.8416	34.2883
5.6	—12.9181	23.5540	20.9626	21.0390
6.8	11.9799	27.5048	30.6335	32.3505

in the determination of the displacements only on the basis of the fourth approximation. In accordance with (2.5), the approximation of the deflection curve is then carried out with a a tenth degree polynomial. The $p_0 - f$ relations obtained according to the first, second, third and fourth approximations are shown in Figs. 2 to 5. It is clear from these curves that the third and fourth are practically indistinguishable. A table of values of p_0 is represented to indicate

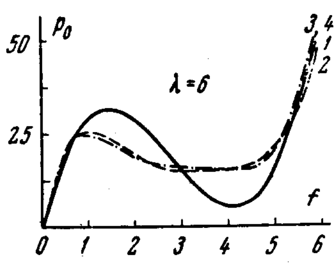

Fig. 4

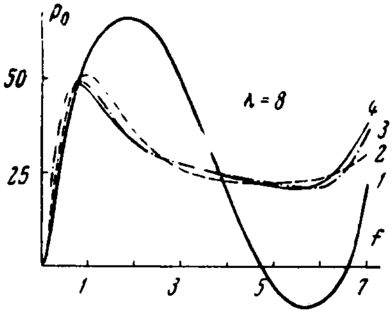

Fig. 5

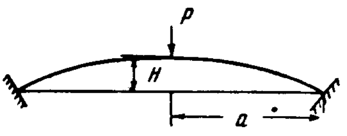

Fig. 6

the rapidity of convergence. It is clear from the table that for $\lambda \leqslant 8$, the fourth approximation differs from the third by an amount of the order of 1.1% even for displacements three to four times the shell thickness. From Figs. 2 to 5, it is obvious that the distinctly expressed tendencies for snap-through of the shell present in the first approximation are smoothed out in the higher approximations. This may be noticed especially beginning with $\lambda = 5$. The disappearance of negative lower critical loads in the higher approximations should also be noted. It may be remarked that as λ increases, the upper and lower critical loads, p^+ and p^-, increase, as is apparent from values given below

$\lambda =$	4	5	6	7	8
$p^+ =$	11.2900	17.0488	24.8538	35.1850	49.5413
$p^- =$	10.1763	12.4166	14.9550	17.8238	20.8838

A simple interpolation formula can be devised which makes possible the direct computation of p^+ and p^- as functions of λ.

$$p^+ = 0.0424\,\lambda^4 - 0.8527\,\lambda^3 + 7.4109\,\lambda^2 - 24.5696\,\lambda + 34.7120 \quad (3.1)$$

$$p^- = 0.0700\,\lambda^4 + 0.1592\,\lambda^3 - 1.1818\,\lambda^2 + 5.7484\,\lambda - 2.3056 \quad (3.2)$$

Equations (3.1) and (3.2) are valid for $\lambda \leqslant 10$. The results of computations for the case of a uniformly distributed load agree very well with the solution of this problem obtained in [1] by a finite difference method.

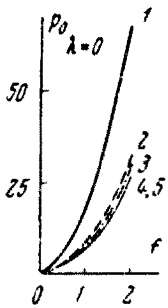

Fig. 7

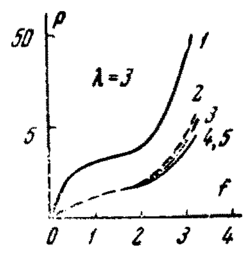

Fig. 8

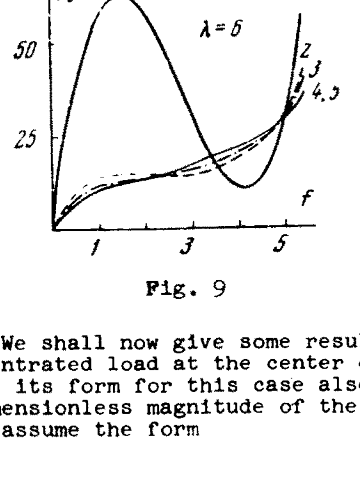

Fig. 9

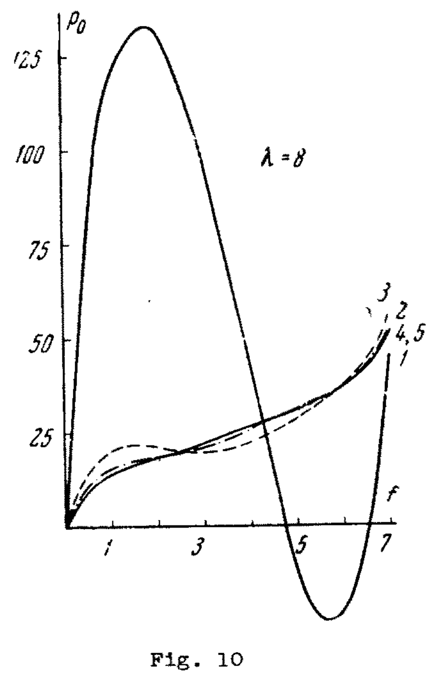

Fig. 10

4. We shall now give some results of an investigation of the effect of a concentrated load at the center of the shell (Fig.6). The system (2.7) retains its form for this case also. However, in this system p_0 is now the dimensionless magnitude of the concentrated force. The corresponding A_r now assume the form

$$A_r = -\frac{3}{\pi (r+1) r} \qquad (4.1)$$

Curves of p_0 vs. f are given for five approximations in Figs. 7 to 10 for $\lambda = 0, 3, 6, 8$. It can be seen from these figures that in this case, as was to be expected, the convergence of the Bubnov-Galerkin method has become slower. This deterioration of the convergence is predicted in a rigorous error analysis of this method [9]. In the present case the difference between the fourth and fifth approximations does not exceed 4% for $\lambda = 8$. It is important to note that in solving the problem in the first approximation, the $p_0 - f$ curves behave the same way as in the case of the distributed pressure. Thus, starting with $\lambda = 4$, characteristic points corresponding to upper and lower critical values are found; i.e. a tendency for snap-through of the shell is revealed. However, in the later approximations this phenomenon disappears and all the way up to $\lambda = 8$ the $p_0 = p_0(f)$ curve is monotonous for the fifth approximation. This last result shows that for loading of the shell by a concentrated force in the center, snap-through does not occur, at least for $\lambda \leqslant 10$.

This conclusion was drawn in [10] on the basis of special assumptions of a geometric character.

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